$\mathrm{stab}_\Gamma(\theta)$ $\mathbb{F}_{p^n}$ K_n D_n E_n S_n $S(n)$ $S_{n,i}$ $S(n,i)$

Morava's Orbit Picture and Morava Stabilizer groups

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L Γ H^* $H_{\mathbb{Z}}L$ Hom $\mathbb{Z}/(p)$ $\mathbb{Z}_{(p)}$ $\mathbb{Z}_p$

Reminders from last time

Lazard Ring $L \cong \mathbb{Z}[x_1, x_2, \dots] \cong MU^*$

Universal group law $G(x,y)$ over L s.t $\forall$ formal group law F over R, $\exists!\ \theta : L \to R$ s.t $F(x,y) = \sum_{i,j} \theta(a_{ij}) x^i y^j$. ($a_{ij}$ are the coefficients of the universal formal group law)

$\Gamma = (\{x + b_1x^2 + b_2x^3 + \cdots \mid b_i \in \mathbb{Z}\}, \circ)$

$\Gamma \curvearrowright L$, $\gamma \longmapsto \theta_\gamma$ induced by $\gamma^{-1}G(\gamma(x), \gamma(y))$

$\log_F$ of a formal group law is a power series s.t

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y)$$

$[n](x) = F(x, [n-1](x))$ with $[1](x) = x$.

height $F(x,y)$ formal group law has height h if
$$[p](x) = ax^{p^h} + (\text{higher terms})$$
w/ a invertible.

v_n given p, coefficient of x^{p^n} in $G(x,y)$

$\mathcal{C}\Gamma$ category of finitely presented, graded L-modules w/ a compatible Γ action.

Class-o-formal groupLaws Two formal group laws over $\mathbb{F}_p$ are isomorphic $\Leftrightarrow$ same height.

Invariant prime ideal theorem

The only prime ideals in L which are invariant under $\Gamma \curvearrowright L$ are $I_{p,n} := (p, v_1, \ldots, v_{n-1})$ where p is prime and $0 \leq n \leq \infty$.

$I_{p,0} = (0)$

$I_{p,\infty} = (p, v_1, \cdots)$

Moreover in $L/I_{p,n}$ for $n > 0$, the subgroup fixed by Γ is $\mathbb{Z}/(p)[v_n]$.

In L, the invariant subgroup is $\mathbb{Z}$.

Landweber Filtration Theorem

Every module M in $\mathcal{C}\Gamma$ admits a finite filtration by submodules in $\mathcal{C}\Gamma$

$$0 = F_0 M \subsetneq F_1 M \subsetneq \cdots \subsetneq F_n M = M$$

such that

each $F_i M / F_{i-1} M \cong$ a suspension of $L/I_{p,n}$ for some p, n.

Takeaway Can localize at p and study

$V_p = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ aka BP_*

Chapter Four

4.1 The action of Γ on L.

Notation Let $H_{\mathbb{Z}} L := \mathrm{Hom}_{Ring}(L, \mathbb{Z})$

Lazard's thm

Definition An automorphism of a formal group law F is a power series $f(x)$ satisfying

$$f(F(x,y)) = F(f(x), f(y)).$$

It is strict if it has the form $x + O(x^2)$

$$f(x) = x + \sum_{i \geq 2} a_i x^i$$

Prop 4.1.1 Let $\Gamma \curvearrowright H_{\mathbb{Z}} L$ be the action induced by $\Gamma \curvearrowright L$.

① $H_{\mathbb{Z}} L \cong \{\text{Formal group laws over } \mathbb{Z}\}$

$\theta : L \to \mathbb{Z}$

$\theta(x_i)$ for each i

② $F, G \in H_{\mathbb{Z}} L$ are in same Γ orbit iff $F \cong G$ over $\mathbb{Z}$.

③ $\mathrm{stab}_{\Gamma}(\theta) =$ strict automorphism group of $\theta \in H_{\mathbb{Z}} L$.

④ Strict automorphism groups of isomorphic formal group laws are conjugate in Γ.

Classification of formal group laws over $\mathbb{Z}$ is tough, but we classified them over $k := \overline{\mathbb{F}_p}$.

Redo!

Let $H_k L := \mathrm{Hom}_{Ring}(L, k)$., Γ_k power series

Prop 4.1.2 The formal group law over k corresponding to $\theta \in H_k L$ has height n if and only if $\theta(v_i) = 0$ for $i < n$ and $\theta(v_n) \neq 0$.

coeff in $G(x,y)$ which is a formal group law over L

Moreover, each $v_n \in L$ is indecomposable, i.e is a unit multiple (in $\mathbb{Z}_{(p)}$) of

$$x_{p^n-1} + \text{decomposables}.$$

4.2 Morava Stabilizer Groups

The nth Morava stabilizer group S_n is the strict automorphism group of a height n formal group law over $K = \overline{\mathbb{F}_p}$

$f(F(x,y)) = F(f(x), f(y))$

It is contained in a division algebra D_n over the p-adic numbers $\mathbb{Q}_p$. We'll get there.

Recall

- $\mathbb{F}_{p^n} = \mathbb{F}_p[\bar{\xi}]$ where $\bar{\xi}$ is a (p^n-1)st root of 1. $\bar{\xi}^{p^n-1} = 1$
- $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic of order n generated by Frobenius auto $x \longmapsto x^p$.
- There is a degree n extension of the p-adic integers $\mathbb{Z}_p$, which we denote $W(\mathbb{F}_{p^n})$ by adjoining a (p^n-1)st root-o-1, ξ where $\equiv \bar{\xi} \bmod p$
- The Frobenius automorphism has a lifting σ, which fixes $\mathbb{Z}_p$, $\sigma(\xi) = \xi^p$ and $\sigma(x) \equiv x^p \bmod p$. $\forall x \in W(\mathbb{F}_{p^n})$

- The fraction field of $W(\mathbb{F}_{p^n})$ is denoted K_n.

- Let $K_n\langle S\rangle$ be K_n adjoined with a noncommuting power series variable S where

$$Sx = \sigma(x)S.$$

$$\zeta^{p^n-1} = 1$$
$$\zeta^{p^n} = \zeta$$

$$S^n x = \sigma^n(x) S^n = x^{p^n} S^n = x S^n$$

(Note S commutes with $\mathbb{Q}_p \subset K_n$ and S^n commutes with everything)

- The division algebra $D_n := K_n\langle S\rangle / (S^n - p)$.

Note: This is a rank n^2 algebra over $\mathbb{Q}_p$ with center $\mathbb{Q}_p$. Rav86 6.2.12

- $E_n := W(\mathbb{F}_{p^n})\langle S\rangle / (S^n - p) \subseteq D_n$.

E_n is a complete local ring with maximum ideal (S) and fraction field $\mathbb{F}_{p^n}$.

Every $a \in E_n$ can be written ! as

$$a = \sum_{i=0}^{n-1} a_i S^i, \quad a_i \in W(\mathbb{F}_{p^n}).$$

OR

$$a = \sum_{i\geq 0} e_i S^i$$ with $e_i \in W(\mathbb{F}_{p^n})$.

where $e_i^{p^n} - e_i = 0$

$e_i = 0$ or a root-of-1.

$$E_n^\times = \left\{ \sum_{i\geq 0} e_i S^i \,\middle|\, e_0 \neq 0 \right\} \text{ or } \left\{ \sum_{i=0}^{n-1} a_i S^i \,\middle|\, a_0 \in W(\mathbb{F}_{p^n})^\times \right\}.$$

Proposition 4.2.5 The full automorphism group of a formal group law over k of height n is isomorphic to $E_n^\times$.

The strict automorphism group S_n is isomorphic to the subgroup

$$\left\{ 1 + \sum_{i>0} e_i S^i \in E_n^\times \,\middle|\, e_i^{p^n} - e_i = 0 \right\} \leq E_n^\times$$

Consider each $e_i : S_n \xrightarrow{\text{cts}} \mathbb{F}_{p^n}$.
The ring of all such functions is

$$S(n) := \mathbb{F}_{p^n}[e_1, e_2, e_3, \dots] / (e_i^{p^n} - e_i)$$

this is a Hopf algebra over $\mathbb{F}_{p^n}$ with coproduct induced by S_n.

Compare to Morava K-theory

$$\Sigma(n) = K(n)_*[t_1, t_2, \dots] / (t_i^{p^n} - v_n^{p^i - 1} t_i)$$

and then $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_{p^n}$. $\left(\begin{matrix} t_i \mapsto e_i \\ v_n \mapsto 1 \end{matrix}\right)$

Lets see how S_n (strict automorphisms of formal group laws) acts on a formal group law of height n, F_n.

Making F_n Let F be a formal group law over $\mathbb{Z}_{(p)}$ with

$$\log_F(x) = \sum_{i \geq 0} \frac{x^{p^{in}}}{p^i}.$$

Then F_n is obtained by reducing F mod p and
- tensoring with $\mathbb{F}_{p^n}$.

An automorphism e of F_n is a power series

$$e(x) \in \mathbb{F}_{p^n}[[x]] \text{ satisfying}$$

$$e(F_n(x,y)) = F_n(e(x), e(y)).$$

Soooo.... given $e = 1 + \sum_{i>0} e_i S^i \in S_n$

$$e(x) = \sum_{i \geq 0}^{F_n} e_i x^{p^i} = F(e_0 x, F(\ldots (\ldots$$

(Notation $x +^F y = F(x,y)$)

See more in Rav 86 Appdx 2.

4.3 Cohomological Properties of S_n.

(to be used by Arseniy and Shangjie later on).

(3 big theorems)

Note: Γ is essentially the multiplication in MU theory. Similarly, S_n is essentially the multiplication in Morava-K-theory.

How?

$FK(n)_*(X) := K(n)_*(X) \otimes_{K(n)_*} \mathbb{F}_{p^n}$

↑ a $K(n)_*$ module w/ $v_n \to 1$.

The group of multiplicative operations here is precisely S_n

Rmk
- $\mathbb{F}_{p^n}$ is essential. (larger or smaller, we lose the result)
- Topology of S_n matters in computation. You can think of S_n as a p-adic Lie group Laz65

Let $H^*(S_n)$ denote the mod p cohomology of S_n and check out Rav86 Chapter 6 for more.

Theorem 4.3.2

a) $H^*(S_n)$ is a finitely generated algebra,

b) If $p-1 \nmid n$, then $H^i(S_n) = \begin{cases} 0 & i > n^2 \\ H^{n^2-i}(S_n) & 0 \le i \le n^2 \end{cases}$ Poincaré Duality

c) If $p-1 \mid n$, then $H^*(S_n)$ is periodic. i.e $\exists x \in H^{2i}(S_n)$ for some $i > 0$ s.t $H^*(S_n)$ is a f.g free module over $\mathbb{Z}/(p)[x]$.

d) Every suff. small open subgroup is cohomologically abelian i.e same cohomology as $\mathbb{Z}_p^{n^2}$.

E_2

Theorem 4.3.3 Let $S_{n,i} \subset S_n$, $i \geq 1$ be the subgroup of $E_n^\times$ that is $\equiv 1 \mod (S)^i$.

i) $S_{n,i}$ are cofinal in the set of open subgroups of S_n

ii) The ring of cts. $\mathbb{F}_{p^n}$-valued functions is

$$S(n,i) = S(n) / (e_j)_{j<i}$$

iii) If $i > \frac{pn}{2p-2}$, the cohomology of $S_{n,i}$ is an exterior algebra on n^2 gen's

iv) Each $S_{n,i}$ is open and normal and $[S_{n,i} : S_{n,i+1}] = p^{ni}$ and $S_{n,i}/S_{n,i+1}$ is abelian.

Theorem 4.3.4 All finite abelian subgroups of S_n are cyclic. S_n contains an element of order p^{i+1} iff $p^i(p-1) \mid n$.

I hope you have
Mor-a-va picture
of these things.

Thanks!

Algebra $A \otimes A \longrightarrow A$

coalgebra $A \xrightarrow{\text{comult}} A \otimes A$